# Persistent algebras

Jérémy S. Cochoy

INRIA Paris-Saclay | jeremy.cochoy@gmail.com

December 2017

How to endow persistent modules with an algebra structure.

1. Persistent modules
2. Homology and cup product
3. Extension of the cup product to a whole module

# This story start with a point cloud

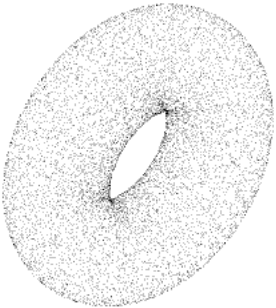


Figure – A set of points with some underlying interesting topology.

# Let $X_t$ be a filtration...

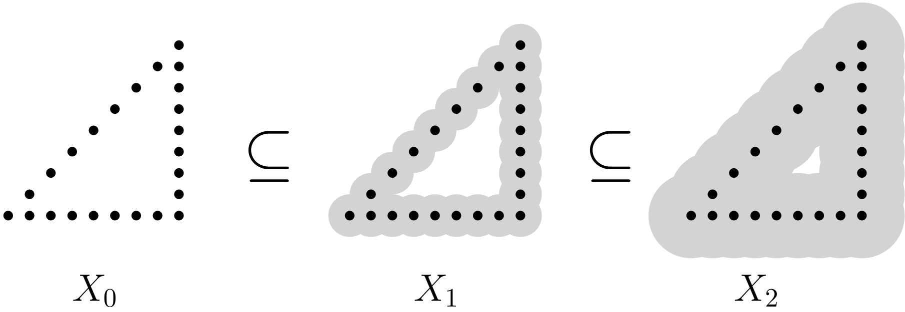



Figure – A filtration

# A bit of (co)homological magic wand

$$H^*(X_0) \xrightarrow[\rho_0^1]{} H^*(X_1) \xrightarrow[\rho_1^2]{} H^*(X_2) \xrightarrow[\rho_2^3]{} H^*(X_3) \xrightarrow[\rho_3^4]{} \dots$$

Figure – A persistence module on $\mathbb{N}$

# The module structure

## Module structure over $k[x]$ :

Let

$$M = \bigoplus_{n \in \mathbb{N}} H^*(X_n)$$

and define the multiplication by elements of $k[x]$ by

$$\forall u \in M, u \in H^*(X_n),$$

$$x.u = \rho_n^{n+1}(u)$$

# The module structure

## Module structure over $k[x]$ :

Let

$$M = \bigoplus_{n \in \mathbb{N}} H^*(X_n)$$

and define the multiplication by elements of $k[x]$ by

$$\forall u \in M, u \in H^*(X_n),$$

$$x.u = \rho_n^{n+1}(u)$$

# Multimodules

## Bifiltrations

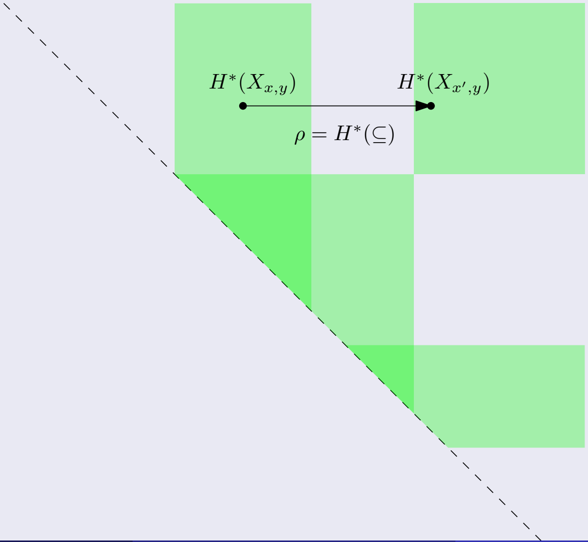
We can build a bifiltration on $\mathbb{R}^2$ :

$$\forall x' \geq x, y' \geq y, X_{x,y} \subseteq X_{(x',y'}.$$

and apply the (co)homology functor.

# Multimodules

## Bimodule $X_{x,y} = f^{-1}([x, y])$



# Nature is wild

The Theory of Multidimensional Persistence
Gunnar Carlsson & Afra Zomorodian

# Singular homology

## Singular simplex

$X$ a topological. $\Delta^n$ the $n$ standard simplex.
A *singular n-simplex* is a continuous map $\sigma : \Delta^n \to X$.

## Chains

$C_n(X)$ free abelian group with basis the singular $n$-simplices.

## Cochains

$C^n(X) = Hom(C_n(X), k)$.

## The cochains $C^n(X) = Hom(C_n(X), k)$ form a chain complex

$\delta^n : C^n(X) \to C^{n+1}(X)$ with

$$\delta\varphi(\sigma) = \sum_i (-1)^i \varphi(\sigma|[v_0, \ldots, \hat{v}_i, \ldots, v_k])$$

The $n$-th cohomology group is $H^n(X) = \operatorname{Ker} \delta^n / \operatorname{Im} \delta^{n-1}$.

## Homology module

$$H^*(X) = \bigoplus_{i \in \mathbb{N}} H^i(X)$$

# Singular homology

## Cup product

Figure – Artistic representation of a cup product

# Cup product

## Product

Let $\varphi \in C^k(X)$ and $\psi \in C^l(X)$.

$$\varphi \smile \psi(\sigma) = \varphi(\sigma|[v_0, \dots, v_k])\psi(\sigma|[v_k, \dots, v_{k+l}]) \in C^{k+l}(X).$$

## Cup product

This product induce a product to the cohomological level : the cup product.

## Graded commutative

For $\alpha \in H^k(X)$ and $\beta \in H^l(X)$,

$$\alpha \smile \beta = (-1)^{kl} \beta \smile \alpha.$$

# Cup product

## Product

Let $\varphi \in C^k(X)$ and $\psi \in C^l(X)$.

$$\varphi \smile \psi(\sigma) = \varphi(\sigma|[v_0, \dots, v_k])\psi(\sigma|[v_k, \dots, v_{k+l}]) \in C^{k+l}(X).$$

## Cup product

This product induce a product to the cohomological level : the cup product.

## Graded commutative

For $\alpha \in H^k(X)$ and $\beta \in H^l(X)$,

$$\alpha \smile \beta = (-1)^{kl} \beta \smile \alpha.$$

# Cup product

## Product

Let $\varphi \in C^k(X)$ and $\psi \in C^l(X)$.

$$\varphi \smile \psi(\sigma) = \varphi(\sigma|[v_0, \dots, v_k])\psi(\sigma|[v_k, \dots, v_{k+l}]) \in C^{k+l}(X).$$

## Cup product

This product induce a product to the cohomological level : the cup product.

## Graded commutative

For $\alpha \in H^k(X)$ and $\beta \in H^l(X)$,

$$\alpha \smile \beta = (-1)^{kl} \beta \smile \alpha.$$

# Cohomology over k

The cohomology is a functor $H^* : \mathbf{Top} \to k$-algebra.

$$H^*(X) = \bigoplus_{d \in \mathbb{N}} H^d(X)$$

# Two bifiltration with same persistent bimodule

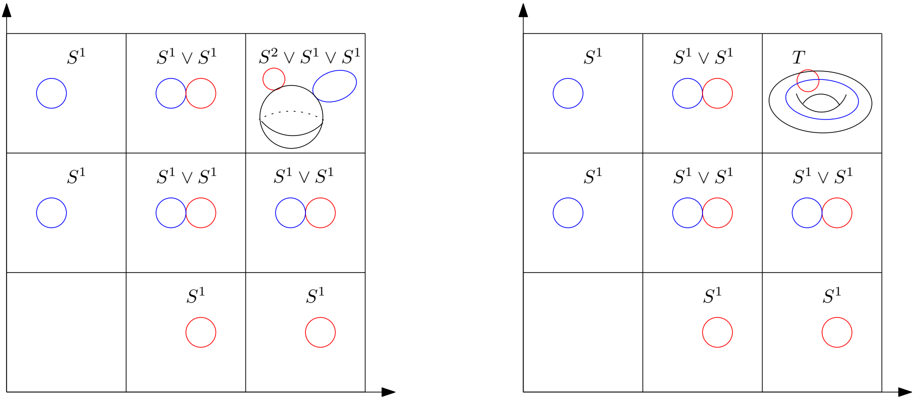



Figure – Two topologycal filtration which give the same persistent bimodule.

Can we extend the cup product to persistence modules ?

$$M = \bigoplus_{(x,y)\in\mathbb{R}^2} \bigoplus_{d\in\mathbb{N}} H^d(X_{x,y})$$

YES ! WE CAN !

Can we extend the cup product to persistence modules ?

$$M = \bigoplus_{(x,y)\in\mathbb{R}^2} \bigoplus_{d\in\mathbb{N}} H^d(X_{x,y})$$

**YES ! WE CAN !**

## Product of two homogeneous vectors

$\forall t, s \in \mathbb{R}^2$ define $t \vee s = (\max(t_x, s_x), \max(t_y, s_y))$.
Let $m \in H^*(X_s), n \in H^*(X_t)$,

$$m \smile n = (x^{s \vee t - s}.m) \smile (x^{s \vee t - t}.n)$$

## Cup product in M

Let $\alpha = \sum m_s, \forall s, m_s \in M_s$, $\beta = \sum n_t, \forall t, m_t \in M_t$. The cup product of this two elements is

$$\alpha \smile \beta = \sum_s \sum_t (x^{s \vee t - s}.m_s) \smile (x^{s \vee t - t}.n_t)$$

## Proposition :

$(M, \smile)$ is a graded-commutative (non-unital) ring.

## Product of two homogeneous vectors

$\forall t, s \in \mathbb{R}^2$ define $t \vee s = (\max(t_x, s_x), \max(t_y, s_y))$.
Let $m \in H^*(X_s), n \in H^*(X_t)$,

$$m \smile n = (x^{s \vee t - s}.m) \smile (x^{s \vee t - t}.n)$$

## Cup product in M

Let $\alpha = \sum m_s, \forall s, m_s \in M_s$, $\beta = \sum n_t, \forall t, m_t \in M_t$. The cup product of this two elements is

$$\alpha \smile \beta = \sum_s \sum_t (x^{s \vee t - s}.m_s) \smile (x^{s \vee t - t}.n_t)$$

## Proposition :

$(M, \smile)$ is a graded-commutative (non-unital) ring.

## Product of two homogeneous vectors

$\forall t, s \in \mathbb{R}^2$ define $t \vee s = (\max(t_x, s_x), \max(t_y, s_y))$.
Let $m \in H^*(X_s), n \in H^*(X_t)$,

$$m \smile n = (x^{s \vee t - s}.m) \smile (x^{s \vee t - t}.n)$$

## Cup product in M

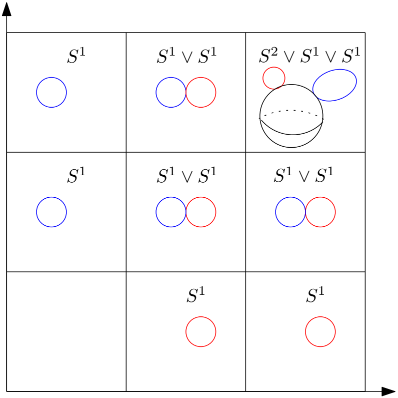
Let $\alpha = \sum m_s, \forall s, m_s \in M_s$, $\beta = \sum n_t, \forall t, m_t \in M_t$. The cup product of this two elements is

$$\alpha \smile \beta = \sum_s \sum_t (x^{s \vee t - s}.m_s) \smile (x^{s \vee t - t}.n_t)$$

## Proposition :

$(M, \smile)$ is a graded-commutative (non-unital) ring.

# Two bifiltration with different persistent algebra



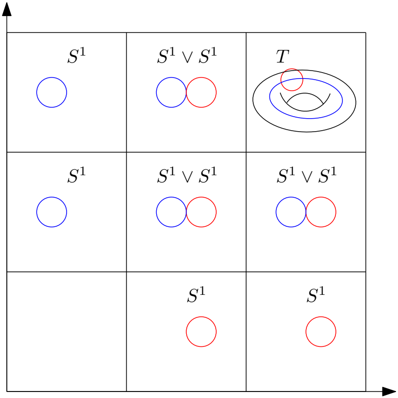




Figure – Two topologycal filtration which give the same persistence bimodules, but two different persistent algebra.

Thanks for your attention.